THE LAPLACE TRANSFORM SERIES DECOMPOSITION METHOD FOR SOLVING NONLINEAR VOLTERRAINTEGRO-DIFFERENTIAL EQUATIONS

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Abstract: The Nonlinearity of differential equations has been a hard to crack nut for years. Recently many researchers have devised several approximate methods of handling it. In this work the Laplace Transform Series Decomposition Method (LTSDM) for solving nonlinear VolterraIntegro Differential Equation is presented. The method is based on the elegant combination of Laplace Transform method, Series expansion method and Adomian polynomial. The numerical results obtained in this work are favourablycompared with the exact solutions and the Modified Homotopy Perturbation Method (MHPM). The compared results clearly showed that the LTSDM is a powerful, accurate, reliable and efficient method.

Keywords: Adomian polynomial, homotopy perturbation, Laplace, nonlinear volterra, series expansion

Introduction

An integro-differential equation is defined by Volterra (1959) and Filiz (2013) as a functional equation in which the unknown function appears in the form of its Derivative as well as under the integral sign. Volterra (1931) studied the hereditary influences when he was examining a population growth model. The research work resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations was termed as Volterraintegro-differential equations.

The general nonlinear Volterraintegro-differential equation is given as;

$$y^{(n)}(x) = f(x) + \int_0^x k(x,t)\left[R\big(y(t)\big) + N\big(y(t)\big)\right] \qquad (1)$$

$$y^{(k)}(0) = b_k, \quad 0 \le k \le (n-1)$$

Where: $y^{(n)}(x)$ is the nth derivative of the unknown function y(x) that is to be determined, k(x,t) is the kernel of the integral equation, f(x) is a known analytic function, R(y) and N(y) are linear and nonlinear functions, respectively. For n=0, (1) turns out to be a classical nonlinear integral equation.

Any Volterraintegro-differential equation is characterized by the existence of one or more of the derivatives $y'(x)$, $y''(x)$, outside the integral sign. The Volterraintegro-differential equations may be observed when we convert an initial value problem to an integral equation by using Leibnitz rule.

Based on the nature of the equation, Integro-differential equations are usually difficult to solve analytically so it is required that we seek an efficient approximate solution. Few out of several numerical methods for approximating the Fredholm or volterra-integro-differential equations are discussed below. Bahugunaet al (2009) examine a comparative study of numerical methods for solving anintegro-differential equation using Laplace, Abdul-Majidwazwaz (2010) applied the combined Laplace transform–Adomian decomposition method for handling nonlinear Volterraintegro–differential equations, Manafianheris (2012) applied Modified Laplace Decomposition to integro differential equation, Majid (2015) introduce a new algorithm for higher order integro-differential equations called modified Laplace decomposition,Sepehrian and Razzaghi (2004) proposed Single-term Walsh series method for solving volterraintegro-differential equations, Brunner (1982) applied a collocation-type method to Volterra-Hammerstein integral equation as well as integro-differential equations. Compact finite difference method has been used for integro-differential equations by Zhao and Corless (2006) while Yalcinbas (2002), Akyaz and Sezer (1999) and Avudainayagam and Vani (2000) used Taylor series, Chebyshev collocation and Wavelet-Galerkin methods, respectively to obtained the solution of integro-differential equation.

In recent years, the application of homotopy perturbation method (HPM) and it's modification (MHPM) in nonlinear problems has been developed by scientists and engineers (He, 2004, 2005a, 2005b; Afrouzi *et al.*, 2011), because this method reduces the difficult problem under study into a simple problem which is easy to solve. Most perturbation methods assume that a small parameter exists, but most nonlinear problems have no small parameter at all. Therefore many new methods, such as the variational method by Liu (2004, 2005), variational iterations method by He (1998a, 1998b), are proposed to eliminate the shortcoming arising in the small parameter assumption. A review of recently developed nonlinear analysis methods can be found in He (2000). In this paper, we propose the use of Laplace Transform Series Decomposition Method to solve both second and fourth order nonlinear Volterraintegro-differential equations and the comparisons of results obtained by LTSDM are made with the exact and modified homotopy perturbation method.

The Method Formulation

Consider the nth order integro-differential equation of the form;

$$y^{(n)}(x) = f(x) + \int_0^x k(x,t)\left[R\left(y(t)\right) + N\left(y(t)\right)\right] \quad (2)$$

With the initial conditions

$$y^{(k)}(0) = b_k, \quad 0 \le k \le (n-1) \quad (3)$$

Assuming $f(x)$ has a series expansion, finds its series expansion and then applies the Laplace transformation on both sides of (2)

$$L[y^n(x)] = L\left[f(x) + \int_0^x k(x,t)\left[R\left(y(t)\right) + N\left(y(t)\right)\right]\right], \quad (4)$$

Using the differentiation property of the Laplace transform, we have

$$s^n L[y(x)] - D^{(n-1)}(y)(0) - sD^{(n-2)}(y)(0) - \ldots - s^{n-1}(y)(0) = L\left[f(x) + \int_0^x k(x,t)\left[R\left(y(t)\right) + N\left(y(t)\right)\right]\right] \quad (5)$$

Further simplification of (5) resulted into

$$L[y(x)] = \frac{1}{s^n}\left(L\left[f(x)\right] + D^{(n-1)}(y)(0) + sD^{(n-2)}(y)(0) + \ldots + s^{n-1}(y)(0)\right) + \frac{1}{s^n}L\left[\int_0^x k(x,t)\left[R\left(y(t)\right) + N\left(y(t)\right)\right]\right] (6)$$

The standard Adomian Decomposition method defines the solution $y(x)$ by the series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (7)$$

and the nonlinear term is decompose as

$$Ny(x) = \sum_{n=0}^{\infty} A_n(y) \quad (8)$$

Where

A_n are the special polynomials called the Adomian polynomials of y_0, y_1, y_2, $y_3, \ldots, y_n$ define by Wazwaz in [11] as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, n = 0,1,2,3\ldots \quad (9)$$

Substitute (7) and (8) into (6) to have

$$\sum_{n=0}^{\infty} y_n(x) = \frac{1}{s^n}\left(L\left[f(x)\right] + D^{(n-1)}(y)(0) + \ldots + s^{n-1}(y)(0)\right) + \frac{1}{s^n}L\left[\int_0^x k(x,t)\left[R\left(y(t)\right) + \sum_{n=0}^{\infty} A_n(y)\right]\right] \quad (10)$$

Using the condition in (3), recursive relation $L[y_0(x)], L[y_1(x)], L[y_2(x)], \ldots$ are obtained.

Taking the Laplace inverse of the recursive relation obtained resulted into the general solution;

$$y = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \ldots \quad (11)$$

Numerical Application

Problem 1:

$$y''(x) = \sinh x + \frac{1}{2}\cosh x \sinh x - \frac{1}{2}x - \int_0^x y^2(t)dt, \qquad y(0) = 0, y'(0) = 1 \quad (12)$$

The exact Solution and the LTSDM solution of problem are:

$$Exact = \sinh x$$

$$LTSDM = x + 0.166667x^3 + 0.00083333x^5 + 0.0001984127x^7 + 2.757732 \times 10^{-6}x^9 + \ldots$$

Problem 2:

$$\left.\begin{array}{l} y^{(4)}(x) = f(x) + 3\int_0^x y^3(t)dt, \\ y(0) = y''(0) = 1, \qquad y'(0) = y'''(0) = -1 \\ f(x) = e^{-x} + e^{-3x} - 1 \end{array}\right\} \quad (13)$$

The exact Solution, the LTSDM solution and MHPM solution of problem are:

$$y(x) = e^{-x}$$

$$LTSDM = 1 - x + 0.5x^2 - 0.166667x^3 + 0.08333x^4 - 0.008333x^5 + 0.00138889x^6$$
$$-0.0001984127x^7 + 2.480157 \times 10^{-5} x^8 + 2.204586 \times 10^{-5} x^9 - 3.196649 \times 10^{-5} x^{10} \ldots$$

$$MHPM = 1 - x + 0.5x^2 - 0.166667x^3 + 0.025x^5 - 0.0125x^6$$
$$+0.005357142857x^7 - 0.0015625x^8 + 0.2480158730 \times 10^{-3} x^9 + 0.3224206351 \times 10^{-5} x^{10} \ldots$$

Problem 3:

$$\left.\begin{aligned} & y^{(4)}(x) = f(x) - \int_0^x y(t) y''(t) dt, \\ & y(0) = y'(0) = y''(0) = 1, \qquad y(1) = e \\ & f(x) = e^x - \frac{1}{2} e^{2x} + \frac{1}{2} \end{aligned}\right\} \tag{14}$$

Since (13) is an initial boundary value problem (IBVP), an assumption is made that

$$y'''(0) = k \tag{15}$$

Having obtained the general solution $y(x)$ in terms of k using LTSDM, the boundary condition $y(1) = e$ is imposed on it in order to find the value $k = 1.000309600$

The exact Solution, the LTSDM solution and MHPM solution of problem are:

$$y(x) = e^x$$

$$LTSDM = 1 + x + 0.5x^2 + 0.0416667x^4 + 0.008333x^5 + 0.00138931169x^6$$
$$+0.0001985334072x^7 + 3.017718 \times 10^{-8} x^8 - 0.00001377195254x^9 - 0.000006612499234x^{10} + \ldots$$

$$MHPM = 1 + x + 0.5x^2 + 0.2308342493x^3 - 0.008333334x^5 - 0.003312507633x^6$$
$$-0.0007480180539x^7 - 0.0001374013388x^8 - 0.00002114462372x^9 - 0.08993064548 \times 10^{-8} x^{13} + \ldots$$

Results and Discussion

Table 1: Comparison of the results obtained by LTSDM with the exact for problem 1

x	Exact	LTSDM	LTSDM error
0	0	0	0
0.1	0.10016675100	0.100166751002	$1.987459*10^{-11}$
0.2	0.20133600250	0.201336002503	$3.934540*10^{-11}$
0.3	0.30452029340	0.304520293401	$7.173049*10^{-10}$
0.4	0.41075232580	0.410752325100	$7.14286*10^{-11}$
0.5	0.52109530550	0.521095300100	$5.404094*10^{-09}$
0.6	0.63665358210	0.636653554300	$2.7814286*10^{-08}$
0.7	0.75858370180	0.758583590100	$1.1165811*10^{-07}$
0.8	0.88810598220	0.888105610200	$3.7204427*10^{-07}$
0.9	1.02651672600	1.026515650000	$1.07582143*10^{-06}$
1.0	1.17520119400	1.175198413000	$2.7812686*10^{-06}$

Table 2: Comparison of the results obtained by LTSDM with the Exact and Modified Homotopy Perturbation Method for problem 2

X	Exact	MHPM	LTSDM	MHPM error	LTSDM error
0	1	1	1	0.00000	0.00000
0.04	0.9607894392	0.9608106692	0.960789545	$2.12300*10^{-05}$	$1.058*10^{-07}$
0.08	0.9231163464	0.9232854120	0.923118053	$1.690656*10^{-04}$	$1.7066*10^{-06}$
0.12	0.8869204367	0.8874885866	0.886929077	$5.681499*10^{-04}$	$8.6403*10^{-06}$
0.16	0.8521437890	0.8534850921	0.852171094	$1.341303*10^{-03}$	$2.73050*10^{-05}$
0.20	0.8187307531	0.8213405980	0.818797419	$2.609845*10^{-03}$	$6.66659*10^{-05}$
0.24	0.7866278611	0.7911217470	0.786766100	$4.493886*10^{-03}$	$1.382389*10^{-04}$
0.28	0.7557837415	0.7628963355	0.756039847	$7.112594*10^{-03}$	$2.561055*10^{-04}$
0.32	0.7261490371	0.7367334755	0.726585945	$1.058444*10^{-02}$	$4.369079*10^{-04}$
0.36	0.6976763261	0.7127037390	0.698376168	$1.150274*10^{-02}$	$6.998419*10^{-04}$

Table 3: Comparison of the results obtained by LTSDM with the exact and modified homotopy perturbation method for problem 3

X	Exact	MHPM	LTSDM	MHPM error	LTSDM error
0	1	1	1	0.00000	0.00000
0.1	1.105170918	1.105230748	1.105170968	$5.9830*10^{-05}$	$5.0*10^{-08}$
0.2	1.221402758	1.221843785	1.221403165	$4.41027*10^{-04}$	$4.07*10^{-07}$
0.3	1.349858808	1.351209687	1.349860175	$1.350879*10^{-03}$	$1.367*10^{-06}$
0.4	1.491824698	1.494673169	1.491827922	$2.848471*10^{-03}$	$3.224*10^{-06}$
0.5	1.648721271	1.653535684	1.648727480	$4.814413*10^{-03}$	$6.209*10^{-06}$
0.6	1.822118800	1.829034189	1.822129142	$6.915389*10^{-03}$	$1.0342*10^{-05}$
0.7	2.013752707	2.022315475	2.013767812	$8.562768*10^{-03}$	$1.5105*10^{-05}$
0.8	2.225540928	2.234405354	2.225559662	$8.864426*10^{-03}$	$1.8734*10^{-05}$
0.9	2.459603111	2.466171899	2.459619956	$6.568788*10^{-03}$	$1.6845*10^{-05}$
1.0	2.718281828	2.718281826	2.718281828	$2.0*10^{-09}$	0

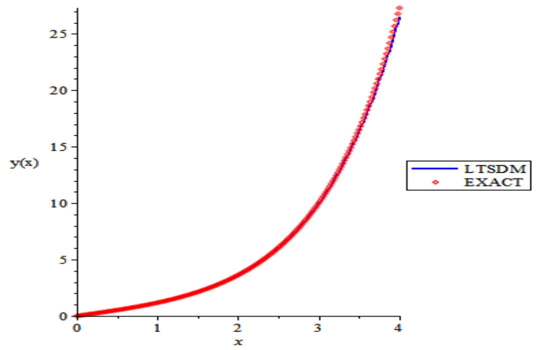

Fig 1: Graphical display of the LTSDM and the exact for the problem 1

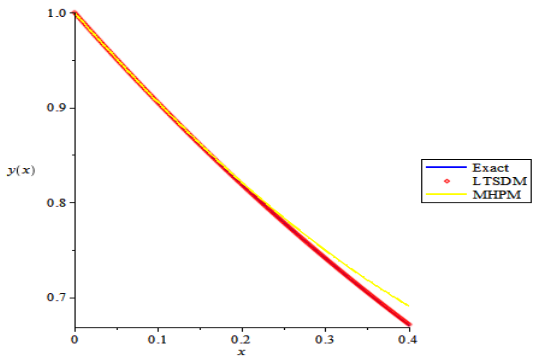

Fig 2: Graphical display of the Exact, LTSDM and the MHPM for the problem 2

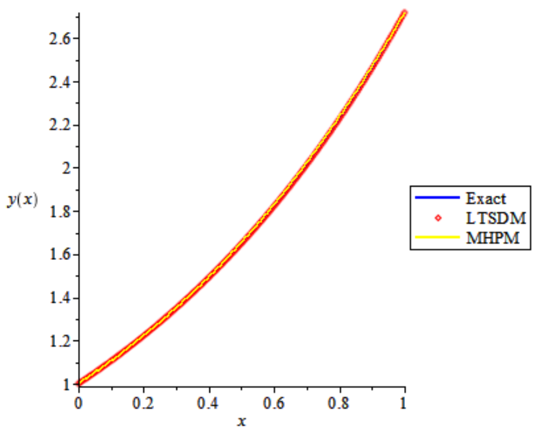

Fig 3: Graphical display of the Exact, LTSDM and the MHPM for the problem 3

Table 1 displayed the numerical results of the exact and LTSDM while Tables 2 and 3 display the comparison of Laplace Transform Series Decomposition Method with the exact and Modified Homotopy Perturbation Method. The error of the results obtained from the three tables show that LTSDM gives a better result than MHPM. Likewise, Figs. 1, 2 and 3 are the graphical representation of the solutions obtained in comparison with the exact and MHPM. It was also discovered from the figures that the LTSDM converged to the exact more rapidly than the Modified Homotopy Perturbation Method.

Conclusion

This work presented an alternative method of solving Nonlinear Volterraintegro-differential equation called Laplace Transform Series Decomposition Method. The method offers significant advantages in terms of its easiness, straightforward applicability, its computational effectiveness and its accuracy. The comparison of the results obtained with the Laplace Transform Series Decomposition Method, the exact and the Modified Homotopy Perturbation method showed that LTSDM gives a better and accurate result than MHPM and at the same time it is capable speeding up the rate of convergence of the solution.

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